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Particle characteristics in a multi-phase Newtonian flow

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Abstract

The flow pattern associated with a multi-phase fluid is the fluid-mediated inter-particle collision, or the exertion of forces on the fluid by the colliding particles. The solid phase is reasonably dispersed but admits collision as a result of either some fluid behaviour or some spatial consideration or both. For some applicable purposes, there may be the need to track a particle's instantaneous position as it is being buffeted during flow. The knowledge of the position, together with a collision history, is critical to predictions concerning the general behaviour of motion. The uncertainty in determining such leads to the recourse to the stochastic differential equation that governs the random stochastic dynamic system - the stable Ornstein-Uhlenbeck process (OUP). A given flow path characterized by the OUP is amenable to the dynamical equation that furnishes the instantaneous position of a particle that is being acted upon by surface forces. Therefore, this work relied on the provision of the equation, whose explicit solution relies on the Fokker-Plank equation that governs the evolution of the space-time probability current.

Keywords and phrases: particles; collision; drag; Weiner process; Markov process

1 Introduction

The flow pattern associated with a single-component fluid is not characterized by fluid-mediated inter-particle collisions or the exertion of forces on the fluid by the colliding particles. This is so because there is no relative motion between the particles and the fluid in the event of flow. When flow is composed of multiple components involving a solid phase and a liquid phase, the associated complexities are so irksome that they require a cautious analysis. There is the convention of modelling the solid-phase as a continuum in keeping with the Eulerian mode of flow regimes. This mode was adopted by Berselli *et al.* [4] wherein the particle-particle interactions were not considered consequential. The use of the Eulerian mode presupposed the minuteness of Stokes' number. In multi-component flow, the fluid is treated as the carrier while the solid-in-motion is treated as the dispersed phase. In Eulerian mode both are treated as interpenetrating fluid media, and therefore both the

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particulate solid-phase and fluid-phase characteristics are represented as a continuous field. In Marble [13] the *Stokes time* was considered small, and the number of particles was assumed reasonably large that the Eulerian approach was seen to be more applicable than Lagrangian mode. Away from Eulerian consideration, there are cases in which the solid-phase is reasonably dispersed but admit collision as a result of either some fluid behaviour or some spatial consideration or both. Mulin [16] discussed the behavior of flows in some spatial configuration, albeit a homogeneous flow. Numerous Lagrangian simulations based on the instantaneous tracking of numerous particles have been conducted. Such may be found in Tanaka *et al.* [21] and in Berlemont *et al.* [3]. Berlemont and Achim [2], adopted the Lagrangian approach in dealing with fluid/solid and solid/solid correlated motion for colliding solid. The importance of inter-particle collision was underscored by Sommerfeld [19] as was earlier indicated by Oesterle´ and Petitjean [17]. Expectedly, inter-particle collisions induce disordered motion, especially a Brownian type (when the size of each particle is considered minute) or when Stokes time is in favour of particle velocity with respect to the surrounding fluid. The spatiotemporal location of a given particle in the milieu of inter-particle collisions is rather stochastic than deterministic. In effect, stochastic differential equations may be used in modelling the behavior of the solid phase when it is subjected to some random excitation. Holmes *et al.* [11] treated an aspect of random process in dealing with coherent structures being mediated by turbulence. Stochastic models for dispersion of flowing particles have been widely used as discussed by Pai and Subramaniam [18] and some references therein. The issue of much importance is the interactions between the dispersed and carrier phases. This engaged the attention of Wells and Stock [24], Liu [12], and Groszmann, and Rogers [10].

Here the surface effect on a two-component fluid is treated. The flow under consideration is laminar. The work sought a stochastic differential equation that governs the random stochastic dynamic system - the stable Ornstein-Uhlenbeck process (OUP) to provide the instantaneous position of a particle that is being acted upon by surface forces. The explicit solution to the equation is dependent upon the provision of the Fokker-Plank equation, which governs the evolution of the space-time probability.

2 The two-component flow model

A multi-component flow model consisting of many disperse (solid) phases and the multi-fluid phase is a regular occurrence in both nature and man-mediated phenomena. The limiting case, which involves a two-component flow is, by far, an easier task to analyze albeit its inherent cumbersomeness. This type of flow is hereunder discussed.

2.1 Transport equation

A generic transport equation is of the form

$$\frac{\partial \varphi}{\partial t}+\frac{\partial (\varphi u_j)}{\partial x_j}+R+\frac{\partial S_j}{\partial x_j}+D+Q=0, \tag{1}$$

where R encodes the radiation effect, D is the dissipation of φ within a volume, V, Q is the production of φ within V and S_j encodes the surface effect.

Assume, first, that the fluid is a single-component one and replace the scalar function φ with the fluid density ϱ. If no relativistic effects prevail, then $D = Q = 0$, and suppose that $R = 0$. The continuity equation reads

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(u_j\rho)=0 \;. \tag{2}$$

Equation (2) above can be used in developing the transport equation of a two-component flow. Denote the two components by ξ (solid) and η (fluid). Then, introduce the mean flow velocity of the components, u^{ξ}_{j}, and u^{η}_{j} respectively, together with their mass concentrations, c^{ξ} and c^{η}. For a two-component flow in which there is no interchange of mass, the continuity equations are of the form

$$\begin{aligned}&\frac{\partial}{dt}(\lambda^{\xi}c^{\xi})+\frac{\partial}{dx_j}(\lambda^{\xi}c^{\xi}u_j^{\xi})=0,\\&\frac{\partial\lambda^{\eta}c^{\eta}}{dt}+\frac{\partial}{dx_j}(\lambda^{\eta}u_j^{\eta}c^{\eta})=0.\end{aligned} \tag{3}$$

In a multi-component flow D and Q in (1) are nonzero. Therefore, the drag force takes the form

$$(D^{\xi\eta}+Q^{\xi\eta})_i=F_i^{\xi\eta} \tag{4}$$

The drag force $F^{\xi\eta}_{i}$ depends on an ensemble of factors. Its dependence on the slip velocity $u_i^{\xi\eta}$, and some other parametric characteristics of the forces exerted on the individual solids in motion is given by

$$F_i^{\xi\eta}=\rho(u_j^{\xi\eta})^2 n_s c_{Di}, \tag{5}$$

where c_{Di} encodes the non-dimensional drag coefficient of the fluid relative to the solids. The numerical value of c_{Di} depends on the dimensionless group applicable, i.e. $c_{Di} = f$ (Re, Fr, We, Ma). In the present case, $c_{Di} = f(\mathrm{Re})$. For spherical solids, the numerical value is [6]

$$c_{Di}=\frac{24}{\mathrm{Re}_{\xi}}\left(1+0.15\mathrm{Re}_{\xi}^{0.687}\right)\quad \text{for } \mathrm{Re}_{\xi}<1000. \tag{6}$$

It is noteworthy to emphasize that the solids considered here are without cavitation. Sometimes would suffice to write the drag force in the form

$$F_i^{\xi\eta}=\frac{\rho^{\xi\eta}u_i^{\xi\eta}}{u_j^{\xi}}g\lambda^{\xi}\lambda_{\eta}^{1-n}, \tag{7}$$

where $u_i^{\xi\eta}$ = (u^{ξ}_{i} -u^{η}_{i}) encodes the local relative velocity of the components, $\rho^{\xi\eta}=(\rho^{\xi}-\rho^{\eta})$; u_j^{ξ} is the final velocity of a single particle in a static fluid and it may be evaluated as

$$u_j^{\xi}=\frac{d_{\xi}^2\rho^{\xi\eta}}{k\mu_{\eta}}g. \tag{8}$$

The expression above serves as a reference for non-uniform-sized particles. The drag force is dependent on the magnitude of the individual particle's relative velocity u^{ξ}_{i} acting in opposite direction and ordered by the particle's Reynold's number

$$\mathrm{Re}_{\xi}=\frac{\rho^{\eta}\left|u_j^{\xi}\right|d^{\xi}}{\mu^{\eta}}, \tag{9}$$

where μ^{η} encodes the viscosity coefficient of the fluid. It is noteworthy that the index n in (7) depends on the particle Reynold number; it decreases as the Reynold number increases (see Glasser *et al.* [8]

and Wallis [22]). The presence of Reynold number (equation (9)) supposes the dominance of the surface forces, especially pressure and viscous stress.

We now suppose the existence of a body force, in which the term R in equation (1) is nonzero. If this force acts on the solid component, then

$$R_j = c^{\xi} \frac{\partial \psi}{\partial x_i}, \tag{10}$$

where ψ encodes the gravitational potential. The surface forces as mentioned may be denoted by $S_{ij}^{\xi\xi}$ and $S_{ij}^{\xi\eta}$. $S_{ij}^{\xi\xi}$ is the effect of inter-particle collisions on the surface and the force exerted by the fluid on the particles is denoted by $S_{ij}^{\xi\eta}$. With these, the momentum equations, for $R, D, Q \neq 0$ each, read

$$\begin{aligned} &\frac{\partial}{\partial t}(\lambda^{\xi} c^{\xi} u_j^{\xi}) + \frac{\partial}{\partial x_j}\{\lambda^{\xi} c^{\xi} (u_j^{\xi})^2\} + \frac{\partial \psi}{\partial x_j} + c^{\xi} \frac{\partial}{\partial x_j}(S^{\Sigma}{}_{ij}) = -\lambda^{\xi} c^{\xi} g + F_i^{\xi\eta}, \qquad (a) \\ &\frac{\partial}{\partial t}(\lambda^{\eta} c^{\eta} u_j^{\eta}) + \frac{\partial}{\partial x_j}\{\lambda^{\eta} c^{\eta} (u_j^{\eta})^2\} + \frac{\partial \psi}{\partial x_j} + c^{\eta} \frac{\partial}{\partial x_j}(S_{ij}^{\eta\eta} + S_{ij}^{\eta\xi}) = -\lambda^{\eta} c^{\eta} g - F_i^{\xi\eta}, \qquad (b) \end{aligned} \tag{11a,b}$$

where λ^{ξ} and λ^{η} are the volume fractions of the solid and the fluid, respectively, g encodes gravity-induced acceleration.

3 Collision effects

Non-constrained particles in flow are amenable to collision largely due to some fluid characteristics and the particles' parameters. The incidence of collision may be on both phases, inducing stress on the disperse phase and an augmented strain rate on the fluid.

3.1 Particle in a stream

In this subsection, the particle(s) being considered are assumed spherical, for ease of analysis. Consider a uniform stream with a stream function $\frac{1}{2}Ur^2 \sin^2\theta$. When a spherical particle, with $r = a$ is inserted into the stream, the stream function reads

$$\psi = \frac{1}{2}Ur^2 \sin^2\theta\left(1 - \frac{a^3}{r^3}\right), \tag{12}$$

and the velocity potential takes the form

$$\varphi = U\left(r\cos\theta + \frac{a^3\cos\theta}{2r^2}\right). \tag{13}$$

If the fluid is at rest at infinity and the sphere moves with the velocity U, then the velocity potential and stream function are [15]

$$\varphi = \frac{1}{2}Ua^3\frac{\cos\theta}{r^2}, \qquad \psi = -\frac{1}{2}Ua^3\frac{\sin^2\theta}{r}. \tag{14}$$

The total kinetic energy of the system fluid and solid) is

$$E_K = \frac{1}{2}(M + \frac{1}{2}M')U^2 \tag{15}$$

where M is the mass of the spherical particle and M' is the mass of the fluid displaced by the particle.

The pressure on the surface of the moving spherical particle reads

$$\frac{p-\Xi}{\rho}=\frac{1}{2}a\mathbf{f}+\frac{1}{8}U^2\left(9\cos^2\theta-5\right),\quad r=a(\text{on the sphere}), \tag{16}$$

where: $\frac{\Xi}{\rho}=\frac{p}{\rho}-\frac{1}{2}\frac{a^3}{r^3}r\mathbf{f}+\frac{1}{8}\frac{U^2a^8}{r^6}\left(3\cos^2\theta+1\right)-\frac{1}{2}\frac{U^2a^3}{r^3}\left(3\cos^2\theta+1\right)$,

$\mathbf{f}=\partial \boldsymbol{U}/\partial t$ (being the acceleration of the centre of the spherical particle).

3.1.2 Solid-phase collisions

An elastic collision is conservatively a compression process that is accompanied by a restitution phase. Upon interaction, the solid deforms and a part of the kinetic energy is stored as strain energy. Then, some remnant of the kinetic energy propagates into the body with compression, shear, or surface Rayleigh waves. For ease of analysis, but without loss of generalization, we shall consider two-particle classes whose diameters are d_1 and d_2 and radii r_1 and r_2 respectively. It is presupposed that the particles are elastic and isotropic. Two scenarios come into play- the case in which the particles are sufficiently close to each other and the case in which the particles are sufficiently isolated. In the first case $S_{ij}^{\xi\xi}$ dominates. Let the instantaneous velocities $\boldsymbol{V}_i^1$ and $\boldsymbol{V}_i^2$, and the number of particles per unit of volume N_{p1} and N_{p2}. The collision frequency with which particle 1 collides with particle 2 is in the form [2]

$$freq_{coll}=\frac{\pi}{4}(d_1+d_2)^2N_{p2}\int\int_{-\infty}^{\infty}\left|\boldsymbol{V}_{j,rel}\right|f_\xi^2d\boldsymbol{u}_1d\boldsymbol{u}_2, \tag{17}$$

where $|\boldsymbol{V}_{rel}|=|\boldsymbol{V}_2-\boldsymbol{V}_1|$ and f_ξ^2 encodes the interparticle pair distribution function. Under molecular chaos assumption in which the colliding particle velocities are assumed Independent, each particle velocity assumes a Gaussian distribution of the form

$$f_\xi^{(1)}=\frac{1}{\left(2\pi\bar{u}_\xi^2\right)^{3/2}}\exp\left(-\frac{\boldsymbol{V}_\xi^2}{2\bar{u}_\xi^2}\right), \tag{18}$$

wherein the interparticle pair distribution function is the product of the two distribution functions.

When the particles collide, the stress and the rate of strain in the solid bodies reads

$$S_{ij}^{\xi\xi}=\lambda\delta_{ij}e_{kk}+2\mu e_{ij}, \tag{19}$$

where the constant μ is the coefficient of rigidity of the solid and λ relates with μ to some parameters.

The collision frequency of the particles is described by Gourdel *et al.* [9] as

$$\mathrm{f}_{\text{coll}}=\frac{\pi}{4}(d_1+d_2)^2N_{\xi 2}\left\|\boldsymbol{V}_{rel}\right\|H(z), \tag{20}$$

where

$$\mathrm{H(z)}=\frac{e^{-z}}{\sqrt{\pi z}}+\mathrm{erf}(\sqrt{z})\left(1+\frac{1}{2z}\right),\quad z=\frac{1}{2}\frac{\left\|\boldsymbol{V}_{\text{rel}}\right\|^2}{\overline{u_1^2}+\overline{u_2^2}}. \tag{21}$$

There are two asymptotic behaviors for large values of z and small values of z. The first case amounts to a high drift between particles, therefore $H(z)$ tends toward unity. Thus, equation (20) reduces to

$$\mathrm{f}_{\text{coll}}=\frac{\pi}{4}(d_1+d_2)^2N_{\xi 2}\left\|\boldsymbol{V}_{rel}\right\|. \tag{22}$$

Therefore, the collision frequency largely depends on particle mean relative velocity. The second case amounts to a small drift between particles. The application of Taylor expansion for $H(z)$, gives [3]

$$\mathrm{f}_{\mathrm{coll}} = \frac{2^{3/2}\pi^{1/2}}{4} N_{\xi 2}(d_1 + d_2)^2 \sqrt{\overline{u_1^2} + \overline{u_2^2}}, \tag{23}$$

which indicates that the collision frequency largely depends on the particle agitation.

If the distance between the particles each time admits frequent collision, then the length of time of application of stress and strain has to be considered. The stress has to be expressed in terms of a memory function (t), which weighs the effect of any previous strain history. Thus,

$$S_{ij}^{\xi\xi}(t) \cong \int_{-\infty}^{t} \beta(t-s) e_{ij}(s) ds. \tag{24}$$

The concern about the motion of any particle with memory may be explained by a generic Langevin equation (see Wang *et al.* [23])

$$\begin{cases} \dot{x} = u & (a) \\ \dot{u} = -\int_0^t \beta(t-t')\gamma(u) u dt' - \nabla U(x) + W(u)\zeta(t) & (b) \end{cases}, \tag{25}$$

where x and u represent a displacement and velocity of the particle, correspondingly. $\beta(t)$ represents the memory function. In (25), $\gamma(u)$ encodes the non-linear damping function, which may be carefully chosen as the procedure suggested by Rayleigh-Helmholtz, $\gamma(u) = u^2 - a$, where a is a positive constant which splits the velocity space into two regions, the positive damping region and the negative driving one. $U(x)$ denotes a generalized potential, taken as $\frac{\omega^2}{2}x^2$ as well as linear -gradient force $\nabla U(x) = \omega_0^2 x$. $W(u)\xi(t)$ encodes random quantity related to the Langevin force induced by fluctuation. $W(u)$ is the strength of noise and $\zeta(t)$ the Gaussian white noise that is categorized by zero mean value, $\langle\zeta(t)\rangle = 0$. $W(u)\ \zeta(t)$ comprises the additive noise $\zeta(t)$ and multiplicative $w(u)\varsigma(t)$($w(u)$ is the strength of multiplicative noise, which may be carefully chosen as the linear function of velocity u). The time correlation function between the memory and the damping term is

$$S_{ij}^{\xi\xi}(t) = \int_0^t [\beta(t-s)e_{ij}(s)]\gamma(u(s))u(s)ds \cdot \tag{26}$$

Wang et al. [19] showed that the memory function in a general form can be written as

$$\beta(t,s) = \frac{\gamma_0}{\Gamma(u)}(t-s)^u \tag{27}$$

where γ_0 encodes the damping coefficient, $u \in (-1,0)$ the power-law exponent, and $\Gamma(u)$ denotes the Gamma function.

3.2 Fluid rate of strain

Tangential stress does not apply on the surface of the fluid at rest on a macroscopic scale. Only the normal stress, $-p\delta_{ij}$, which is thermodynamic in origin, and is sustained by molecular collisions applies. When the fluid is in relative motion on the continuum scale an additional stress τ_{ij} applies (see Mei [14] for elementary theory). The resulting total stress is

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{28}$$

The viscous stress τ_{ij} must depend on velocity gradients,

$$\frac{\partial q_i}{\partial x_j}, \frac{\partial^2 q_i}{\partial x_j \partial x_k}, \ldots \tag{29}$$

The gradient $\frac{\partial q_i}{\partial x_j}$ consists of two parts,

$$\frac{\partial q_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right), \tag{30}$$

with the rate of strain tensor and the vorticity tensor given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i}\right) \quad \text{and} \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}\right) \tag{31}$$

respectively. We note that fluid incompressibility requires that $\nabla.\boldsymbol{q} = 0$, and with the viscous stress tensor

$\tau_{ij} = \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i}\right)$, the total stress is

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i}\right) \quad . \tag{32}$$

In the fluid/fluid interaction regime the surface effect $S_{ij}^{\eta\eta}$ may be determined in terms of the rate of strain in the fluid as

$$S_{ij}^{\eta\eta} \equiv e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{33}$$

3.3 Infinitely large surface effect

We consider flows wherein the disperse-phase is a bit close-packed. In the absence of the fluid, the particle phase may be modeled as a coarse flow based on the kinetic theory description. By the presence of the fluid phase, the particle momentum is modified through a drag term in the kinetic equation. Suppose two spherical particles in flow whose radii are *a* and *b*, and are of distance *c* units apart are sufficiently dispersed. The surface stress $S_{ij}^{\xi\eta}$ dominates. Assume that the particles sustain their fixed position and undergo oscillatory motion about their fixed position. Then the mean value of the force acting on each of the particles is [15]

$$S_{ij}^{\xi\eta} = 3\pi\rho\left(\frac{a^3 b^3}{c^4}\right) kk'p^2 \cos\varepsilon, \tag{34}$$

where *k, k'* are the amplitudes of oscillation, $\frac{2\pi}{\rho}$ the period, and ε the phase difference. When the particle behind is released, and it moves closer to the one in front, the distance *c* decreases. This decrease is progressive as the behind particle gets closer. Thus,

$$\lim_{c \to 0} S_{ij}^{\xi\eta} = \infty \quad . \tag{35}$$

The above explains an infinitely large surface effect occasioned by particle collision as described. It is of note that hydrodynamic coupling, through the drag term, induces particle clustering and cluster-induced turbulence [5] in the fluid–particle flow. Within the same flow field, therefore a turbulent fluid–particle flow has time-dependent regions wherein the particle phase is almost in clusters. Depending on the fluid rheological characteristics, the cluster may be separated by regions wherein the particle–particle collisions are absent.

4 Statistical ensemble

Let us consider particles that are being buffeted during flow, whose temporal location may be determined amid great uncertainty. In the face of such difficulty, a recourse to a stochastic differential equation that governs a random stochastic dynamic system may lead the way. The stable Ornstein-Uhlenbeck process (OUP), is of the form

$$\dot{x} = \upsilon\left(\gamma - x_t\right) + \varepsilon\varphi(t), \tag{36}$$

where υ is a parametric, x_t is the particles' temporal location, and $\varphi(t)$ (the white noise) is the formal derivative of a non-zero mean Weiner process (W_t) (see Holmes and Berzook [11]). It holds well for the motion of a particle in a quadratic potential well subject to first-order dynamics and random excitation (see Arnold [1]). In Einstein's idealization, the description of Brownian motion indicates that increments of a particle's position over disjoint time intervals are independent random quantities. Wiener process describes a probabilistic model in which a random trajectory, although continuous, is nowhere differentiable. Since W_t is nowhere differentiable, $\varphi(t)$ is a distribution rather than a function. $x(t)$ may be represented unconditionally as a scaled time transformed Weiner process in the form

$$h_s(x) = \sqrt{\frac{\theta}{\pi\sigma^2}}e^{-\theta(x-\mu)^2/\sigma^2} \tag{37}$$

or conditionally (given x_0)

$$x_t = x_o e^{-\theta t} + \mu(1-e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}}W(e^{2\theta t})e^{-\theta t}\cdot \tag{38}$$

In a section of flow where a particle obeys the model (36), a probabilistic description of its temporal location may be given as a conditional probability density function (pdf) of the form

$$h\left(x,t \mid x_0\right) \in \left(-\infty,+\infty\right)\times\left[0,+\infty\right), \quad x_0 = x(0). \quad (h \text{ is a pdf}) \tag{39}$$

In equation (39) the first term on the right-hand side encodes the *drift*; if $\upsilon < 0$, the particle drifts along the vector field, and if $\varepsilon \neq 0$ the particle is buffeted by a random force. The explicit solution of equation (36) may be found by recourse to the Fokker-Plank equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[\upsilon(x-\mu)p\right] + \frac{\varepsilon^2}{2}\frac{\partial^2 h}{\partial x^2}\cdot \tag{40}$$

The terms on the right-hand side of (40) above are associated with surface effects, with the first encoding the drift and the second the diffusion. The solution is [1]

$$h(x,t \mid x_0) = \sqrt{x_0 e^{\upsilon t}, \frac{\varepsilon^2}{2\upsilon}\left(e^{2\upsilon t}-1\right)}, \quad x(0) = x_0 \quad , \tag{41}$$

with $\sqrt{(\mu,\sigma^2)} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$ as the normal distribution with mean μ and σ the variance. The stationary solution is a Gau-ssian distribution given by

$$h_s(x) = \sqrt{\frac{\theta}{\pi\sigma^2}}e^{-\theta(x-\mu)^2/\sigma^2} \tag{42}$$

4.2 Particle collision history

The history of collision is a *sine-qua-non* to the making of predictions regarding the general behaviour of motion. In some dynamic phenomena, such as the one being considered, an existing state may be influenced by former states. Specifically, the sequence of previous collisions between two flowing particles, say, may give a clue to the possibility of the existence of some memory, which will in effect ease predictions.

If the present state depends on the former states, there is a describing linear memory equation of the form [20]

$$\dot{u}(t) = -au + K * u = -au + \int_0^t K(t-s)u(s)ds, \quad u(0) = u_0 \tag{43}$$

where u: $[0, \infty) \rightarrow \mathbb{R}$ is a scalar state variable, $u_0 \in \mathbb{R}_{\geq 0}$ and K: $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a positive real kernel. It is observed from the above equation that the second member of the right-hand side (which encodes the prevalence of the previous states) halts the speedy decay of the system.

4.2.1 Collision and Markov process

Definition (1). Suppose $\boldsymbol{\varsigma}$ is a σ-algebra, $\boldsymbol{\varsigma} \subseteq U$, then $P(A \mid \boldsymbol{\varsigma}) := E(\boldsymbol{\chi}_A \mid \boldsymbol{\varsigma})$ for $A \in U$. Thus, $P(A \mid \boldsymbol{\varsigma})$ is the conditional probability of A, given $\boldsymbol{\varsigma}$, and a random variable.
Given a stochastic process X(.), the σ-algebra

$$U(s) := U(X(l) \mid 0 \leq l \leq s \tag{44}$$

is termed the *history of the process* up to and including time s (see Evans [7]). Thus, in the stochastic collision episode, $U(s)$ tracks the statistics accessible from observing $X(l)$ for all times $0 \leq l \leq s$.

Definition (2) [7]. An R^n-valued stochastic process X(·) is called a Markov process if

$$P(X(t) \in B \mid U(s)) = P(X(t) \in B \mid X(s)) \text{ a.s.} \tag{45}$$

for all $0 \leq s \leq t$ and all Borel subset B of R^n.
Suppose the collision process is denominated at present value $X(s)$, the probabilities of future values of $X(t)$ may be predicted in a way similar to the possible knowledge of the total history of the process before time s. Thus, the process only recognizes its value at time s and does not remember how it got there. Markov property, therefore somewhat explains the non-differentiability of particle paths in Brownian motion.

5 Summary and conclusion

In the flow of a single-component fluid, there are no fluid-mediated inter-particle collisions and no exertion of forces on the fluid by colliding particles. Flows involving a solid phase and a liquid phase present associated complexities that are so irksome that they entail a rigorous analysis. The thrust of this work is the examination of surface forces on a two-phase flow. The flow consists of the liquid phase, which is considered a carrier and the disperse phase, whose motion is in the direction of the flowing fluid and whose content (the particles) is in a mutual state of collision. The uncertainty in determining the temporal location of a flowing particle that is being buffeted leads to the recourse to the stochastic differential equation that governs a random stochastic dynamic system - the stable Ornstein-Uhlenbeck process whose equation furnishes the "white noise", term which is the formal derivative of a non-zero mean Weiner process (W_t).

Collision history is a *sine-qua-non* to predictions concerning the general behaviour of motion. Since an existing state may be influenced by former states, the sequence of previous collisions between two flowing particles, say, may give a clue to the possibility of the existence of some memory, thereby aiding predictions. When a given flow path is characterized by the OUP, the equation (41) furnishes the instantaneous position of a particle that is being acted upon by surface forces.

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