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A Lyapunov-based approach to uniform strict stability analysis for nonlinear impulsive Caputo fractional derivatives with weakly singular kernel

J.U. Atsu*, J.E. Ante†‡, C.M. Orazulume§, S.O. Essang¶, R.E. Francis‖, S.I. Okeke**and W.K. Udogworen††

Abstract

This study explores the uniform strict stability of nonlinear impulsive Caputo fractional differential equations (ICFrDEs) by utilizing a powerful framework of vector Lyapunov functions which is generalized by a class of piecewise continuous Lyapunov functions. By integrating comparison results and employing this new class of Lyapunov functions, comprehensive sufficient conditions for uniform strict stability are derived. The applicability and significance of these findings are illustrated through a detailed example, which demonstrates the improvements over existing results and highlights the broader implications of this approach. This research contributes significantly to the analysis of uniform strict stability of ICFrDEs, providing a robust framework for future studies.

Keywords and phrases: uniform strict stability; Caputo derivative; impulse; vector Lyapunov function; fractional differential equation.

1 Introduction

The theory of fractional differential equations (FrDEs) is unarguably an extension or the generalization of the traditional concept of differential equations. It has been adopted as an efficient tool in the modeling of various real life problems and phenomena [1, 2, 3], and moreso, having played an important role in physics, engineering, control theory, dynamical systems, etc., it can be used to model electrical and mechanical properties of materials as well as in the description of properties of gases, liquids and rocks, among other fields (see also [4, 5]).

The earliest study of the concept began about the 19th century during a mathematical discourse that transpired between Leibnitz and L'hopital when the former introduced the notation for derivative to be d^n/dx^n. This notation, however generated lots of questions among mathematicians - the question of whether the validity of the order n of the derivative will hold if it is extended to non-integer. In any case, as a direct question to Leibnitz, L'hopital in his letter asked "what the result would be if $n = 1/2$?" In his response, Leibnitz averred, "it could be an apparent paradox from which one day

*Department of Mathematics, University of Cross Rivers State, Calabar, Nigeria; E-mail: jeremiahatsu@unicross.edu.ng

†Corresponding Author; E-mail: jackson.ante@topfaith.edu.ng

‡Department of Mathematics, Topfaith University, Mkpatak, Akwa Ibom State, Nigeria

§Department of Electrical/Electronics, Topfaith University, Mkpatak, Nigeria; E-mail: mc.orazulume@topfaith.edu.ng

¶Department of Mathematics and Computer Science, Arthur Jarvis University, Akpabuyo, Cross River State, Nigeria; E-mail: sammykmf@gmail.com

‖Department of Statistics, Federal Polytechnic, Ugep, Nigeria; E-mail: runyifrancis@fedpolyugep.edu.ng

**Department of Industrial Mathematics and Health Statistics, David Umahi Federal University of Health Sciences, Uburu, Ebonyi State, Nigeria. E-mail: okekesi@dufuhs.edu.ng

††Department of Mathematics, Akwa Ibom State University, Ikot Akpaden, Mkpat-Enin. E-mail: wisdomudogworen@aksu.edu.ng

useful consequences would be drawn." Since then, research interest in the study of fractional derivatives has experienced a towering blossoming. With this development, it has been observed that the behavior of several systems, including phenomena with memory and hereditary characteristics can be modelled by fractional dynamical systems (see[6, 7, 8]). As an interesting developing area of research in calculus during the last couple of decades, fractional calculus offers an enormous robust characteristics with strong relevance in contemporary applications. For a more detailed discussion on the subject, we would refer interested readers to [9, 10, 11].

Strict stability is an aspect of stability which helps us to ascertain the rate of decay of the solutions. This is due to the fact that - as observed in [12] the usual stability concepts do not give any information about the rate of decay of solutions, albeit it constitutes the main properties in the qualitative theory of differential equations. Due to this lapses, the notion of strict stability has however endeared research interest, given the fact that it can be likened to the usual concept of uniform asymptotic stability. In [13], strict stability results was examined for impulsive differential equations using the Lyapunov functions. Fundamental results for the uniform strict practical stability for impulsive differential system with a delay was examined in [14] using the method of Lyapunov functions together with the Razumikhin technique.

Furthermore, the analysis of the stability properties of fractional differential equations (FDEs) is more complicated than the classical derivatives. This complexity is due to the fact that, fractional derivatives have nonlocal conditions and singular kernels (see [15, 16]). The nonlocality of the fractional derivatives is implied by the notion that the properties of the fractional derivative is dependent on both the initial condition and the order of the derivative. The stability properties of solutions of FrDEs using the scalar Lyapunov function was examined in [17, 18, 19], and sufficient conditions for the stability as well as the uniform stability properties of FrDEs using the vector Lyapunov function was examined in [20, 21, 22].

Moreso, in the analysis of the stability properties of solutions of fractional order systems, one of the viable tool that is often employed is the Lyapunov's second method, also called the Lyapunov's direct method. This method has been argued among researchers to be more versatile in examining the stability properties of solutions compared to other approaches like the monotone iteration method, Laplace Adomian decomposition method, Laplace transform, Razumikhin technique, etc., since it allows us to predict the stability of solutions of differential systems without first solving the given systems (see, for example, [23]). The two main groups of Lyapunov functions have been discussed in [24].

Now, impulsive differential equations (IDEs) have emerged as a powerful and versatile modeling tool, capable of accurately capturing the dynamics of real-world processes and phenomena. In contrast to traditional differential equations, IDEs provide a more comprehensive and realistic framework for describing complex systems in fields such as engineering, physics, chemistry, economics, and finance. The theory of IDEs is particularly well-suited for modeling situations where sudden changes or impulsive effects occur, making it a valuable asset for researchers and practitioners seeking to understand and analyze complex systems [25]. In addition, studies have demonstrated that impulses can have a stabilizing effect on systems that are inherently unstable, allowing for the mitigation of instability and the promotion of equilibrium in complex systems. Thus, it becomes needful to study them.

Many evolutionary processes are marked by sudden and abrupt changes in state, which occur at specific points in time. These changes are so rapid that their duration is often considered insignificant compared to the overall duration of the process. To effectively apply impulsive differential systems, it is essential to develop criteria for assessing the stability of their solutions, which can help predict and understand the behavior of these complex systems [35]. Although Lyapunov's second method is a powerful tool for analyzing stability properties of differential systems, its use is not without restrictions, particularly when it comes to (IDEs), where its applicability is limited. As observed in [36], the traditional approach of using continuous Lyapunov functions restricts the effectiveness of the method, but given that the solutions of the systems being evaluated are piecewise continuous, it is necessary to employ Lyapunov functions that can accommodate discontinuities of the first kind, thereby allowing for a more accurate analysis. The stability properties of impulsive differential system has been examined in ([25, 37]). [23] examined the eventual stability properties of impulsive fractional order systems using the vector form of the Lyapunov functions.

In this paper, the uniform strict stability of ICFrDEs is addressed using the vector form of the Lyapunov function which is generalized by a class of piecewise continuous Lyapunov functions. Together with the comparison results, sufficient conditions for the uniform strict stability of the system is established with an illustrative example.

2 Preliminaries, Definitions and Notations

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with norm $\|.\|$ and let Ω be a domain in $\mathbb{R}^n$ containing the origin; $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $t_0 \in \mathbb{R}_+, t > 0$.
Let $J \subset \mathbb{R}_+$ and define the following class of functions $PC^q[J, \Omega] = \alpha : J \to \Omega, \alpha(t)$ as piecewise continuous mapping with order q from the domain J into the range Ω with points of discontinuity $t_k \in J$ at which $\alpha(t_k^+)$ exists.
Fractional calculus being the generalization of the classical calculus to non integer order allows for the extension of the traditional concepts of derivative and integral to functions with fractional orders. It allows for functions with non integer orders which makes it much more flexible in describing real world systems (see [31]).
There are several definitions of fractional derivatives and fractional integrals

General case

Let the number $n - 1 < q < n, q > 0$ be given, where n is a natural number and $\Gamma(.)$ denotes the gamma function.

1 The Riemann-Liouville fractional derivative of order q of $x(t)$ is given by (see [16])

$$ {}_{t_0}^{RL}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-q-1} x(s) ds, t \geq t_0 $$

2 The Caputo fractional derivative of order q of $x(t)$ is defined by (see [32])

$$ {}_{t_0}^{C}D^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s) ds, t \geq t_0 $$

The Caputo derivatives has many properties that are similar to those of the standard derivatives, which makes them easier to understand and apply. The initial conditions of fractional differential equations using the Caputo derivative are also easier to interpret in physical context, which is another reason why it is often used in applications of fractional calculus.

3 The Grunwald-Letnikov fractional derivative of order q of $x(t)$ is given by (see [23])

$$ D_0^q x(t) = \lim_{h \to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t - rh), t \geq t_0 $$

and the Grunwald-Letnikov fractional Dini derivative of order q of $x(t)$ is given by (see [24])

$$ D_0^q x(t) = \limsup_{h \to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t - rh), t \geq t_0 $$

where ${}^qC_r$ are the binomial coefficients and $[\frac{(t-t_0)}{h}]$ denotes the integer part of $\frac{(t-t_0)}{h}$.

Particular case

(when n=1). In most applications, the order of q is often less than 1, so that $q \in (0,1)$. For simplicity of notation, we will use ${}^CD^q$ instead of ${}^C_{t_0}D^q$ and the Caputo fractional derivative of order q of the function $x(t)$ is

$$ {}^CD^q x = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x' ds, t \geq t_0 \tag{1} $$

3 Impulses in Fractional Order Systems

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}^CD^q x = f(t,x), t \geq t_0, x(t_0) = x_0, \tag{2} $$

where $x \in \mathbb{R}^N$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t,0) \equiv 0$ and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^N$.
Some sufficient conditions for the existence of the global solutions to (2) are considered in [27, 33]. The IVP for FrDE (2) is equivalent to the following Volterra integral equation (See [29]),

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, t \geq t_0 \tag{3} $$

Consider the initial value problem for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0 < q < 1$,

$$ \begin{aligned} {}^CD^q x = f(t,x), t \geq t_0, t \neq t_k, k = 1, 2, ... \\ \Delta x = I_k(x(t_k)), k \in N, t = t_k \\ x(t_0) = x_0, \end{aligned} \tag{4} $$

where $x, x_0 \in \mathbb{R}^N$, $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, and $t_0 \in \mathbb{R}_+$, $I_k : \mathbb{R}^N \to \mathbb{R}^N$, $k = 1, 2, ...$
under the following assumptions:

(i) $0 < t_1 < t_2 < ... < t_k < ...$, and $t_k \to \infty$ as $k \to \infty$;

(ii) $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, is continuous in $(t_{k-1}, t_k]$ and for each $x \in \mathbb{R}^N$, $k = 1, 2, ...,$ $\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$ exists;

(iii) $I_k : \mathbb{R}^N \to \mathbb{R}^N$

In this paper, we assume that $f(t,0) \equiv 0$, $I_k(0) = 0$ for all k, so that we have the trivial solution for (4), and the points $t_k, k = 1, 2, ...$ are fixed such that $t_1 < t_2 < ...$ and $\lim_{k\to\infty} t_k = \infty$. The system (4) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^N)$.

Remark 3.1. The second equation in (4) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 3.2. *Let $V : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N_+$. Then V is said to belong to class L if,*
(i) V is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^N$ and for each $x \in \mathbb{R}^N$, $k = 1, 2, ...$ and $\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x)$ *exists;*

(ii) V is locally Lipschitz with respect to its second argument x and $V(t,0) \equiv 0$.

Now, for any function $V(t,x) \in PC([t_0,\infty)\times\xi, \mathbb{R}^N_+)$ we define the Caputo fractional Dini derivative as:

$$ {}^cD^q_+V(t,x) = \limsup_{h\to 0^+} \frac{1}{h^q}\{V(t,x) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)]\} \tag{5}$$

$t \geq t_0$ where $t \in [t_0,\infty), x, x_0 \in \xi$, $\xi \subset \mathbb{R}^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T]$.

Definition 3.3. *A function $g \in PC[\mathbb{R}^n, \mathbb{R}^n]$ is said to be quasi-monotone non-decreasing in x, if $x \leq y$ and $x_i = y_i$ for $1 \leq i \leq n$ implies $g_i(x) \leq g_i(y), \forall i$.*

Definition 3.4. *The zero solution of (4) is said to be:*
$(SS1)$ strictly stable if for every $\epsilon_1 > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta_1 = \delta_1(\epsilon_1, t_0) > 0$, such that for any $x_0 \in \mathbb{R}^N$, $\|x_0\| \leq \delta_1$ implies $\|x(t;t_0,x_0)\| < \epsilon_1$ for $t \geq t_0$, and for any $\delta_2 = \delta_2(\epsilon_1, t_0)$, $\delta_2 \in (\delta_1, 0]$ there exists $\epsilon_2 = \epsilon_2(t_0,\delta_2), \epsilon_2 \in (0,\delta_2]$ such that $\delta_2 < \|x_0\|$ implies $\epsilon_2 < \|x(t;t_0,x_0)\|$ for $t \geq t_0$.

$(SS2)$ strictly uniformly stable if for every $\epsilon_1 > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta_1 = \delta_1(\epsilon_1) > 0$, such that for any $x_0 \in \mathbb{R}^N$, $\|x_0\| \leq \delta_1$ implies $\|x(t;t_0,x_0)\| < \epsilon_1$ for $t \geq t_0$, and for any $\delta_2 \in (\delta_1, 0]$ there exists $\epsilon_2 = \epsilon_2(\delta_2), \epsilon_2 \in (0,\delta_2]$ such that $\delta_2 < \|x_0\|$ implies $\epsilon_2 < \|x(t;t_0,x_0)\|$ for $t \geq t_0$

Definition 3.5. *A function $a(r)$ is said to belong to the class $\mathcal{K}$ if $a \in PC([0,\psi), \mathbb{R}_+], a(0) = 0$, and $a(r)$ is strictly monotone increasing in r.*

In this paper, we define the following sets:

$$\begin{aligned} \bar{S}_\psi &= \{x \in \mathbb{R}^N : \|x\| \leq \psi\} \\ S_\psi &= \{x \in \mathbb{R}^N : \|x\| < \psi\} \end{aligned}$$

It suffices to say that the inequalities between vectors are understood to be component-wise inequalities.

We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$\begin{aligned} {}^c_{t_0}D^q u = g(t,u), t \geq t_0, t \neq t_k, k = 1,2,... \\ \Delta u = \psi_k(u(t_k)), k \in N, t = t_k \\ u(t_0^+) = u_0, \end{aligned} \tag{6}$$

existing for $t \geq t_0$, where $u \in \mathbb{R}^n, \mathbb{R}_+ = [t_0,\infty), g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, g(t,0) \equiv 0$,
where g is the continuous mapping of $\mathbb{R}_+ \times \mathbb{R}^n$ into $\mathbb{R}^n$. The function $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is such that for any initial data $(t_0,u_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the system (6) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t;t_0,u_0) \in PC^q([t_0,\infty), \mathbb{R}^n)$, (see [30, 33, 38]).

Lemma 3.6. *Assume $m \in PC([t_0,T] \times \bar{S}_\psi, \mathbb{R}^N)$ and suppose there exists $t^* \in (t_0,T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*,\alpha_1) = m(t^*,\alpha_2)$ and $m(t,\alpha_1) < m(t,\alpha_2)$ for $t_0 \leq t < t^*$. Then if the Caputo fractional Dini derivative of m exists at t^*, then the inequality ${}^CD^q_+m(t^*,\alpha_1) -{}^C D^q_+m(t^*,\alpha_2) > 0$ holds.*

Proof. Let $V(t,x) = m(t,\alpha_1) - m(t,\alpha_2)$.
Applying (5), we have

$$\begin{aligned} {}^CD^q_+(m(t^*,\alpha_1) - m(t^*,\alpha_2)) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t^*,\alpha_1) - m(t^*,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ &\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r[m(t^*-rh,\alpha_1) - m(t^*-rh,\alpha_2)] \\ &\quad -[m(t_0,\alpha_1) - m(t_0,\alpha_2)]\} \end{aligned}$$

When $m(t^*, \alpha_1) = m(t^*, \alpha_2)$, we have

$$\begin{aligned}
{}^{C}D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{-[m(t_0, \alpha_1) - m(t_0, \alpha_2)] - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r \\
&\quad [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)]\} \\
&= -\limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad + \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r[m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] \\
&\quad - \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&= -\limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&= -\limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r qC_r[m(t_0, \alpha_1) - m(t_0, \alpha_2)]
\end{aligned}$$

Applying equation 3.8 in [24], we have

$${}^{c}D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) = -\frac{(t - t_0)^{-q}}{\Gamma(1 - q)}[m(t_0, \alpha_1) - m(t_0, \alpha_2)]$$

By the lemma, we have that

$$m(t, \alpha_1) - m(t, \alpha_2) < 0, \text{for}\, t_0 \leq t < t^*$$

And so it follows that

$${}^{c}D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) > 0$$

Remark 3.7. Lemma 3.2 extends Lemma 1 in [24], where the vectors $m(t, \alpha_1)$ and $m(t, \alpha_2)$ are compared component-wise.

4 Fractional Differential Inequalities and Comparison results for Impulsive vector Fractional Differential Equations

In this section, we assume that $0 < q < 1$.

Theorem 4.1. *Assume that*

(i) $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n]$ *and is continuous in* $(t_{k-1}, t_k]$, $k = 1, 2, \ldots$ *and* $g(t, u)$ *is quasimonotone nondecreasing in* u *for each* $u \in \mathbb{R}^n$ *and* $\lim\limits_{(t,y)\to(t_k^+,u)} g(t, u) = g(t_k^+, u)$ *exists;*

(ii) $V \in PC[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N_+]$ *and* $V \in \mathcal{L}$ *such that*
${}^{C}D^q_+ V(t, x) \leq g(t, V(t, x))$, $t \neq t_k$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$ *and*
$V(t_k, x + I_k(x(t_k))) \leq \rho_k(V(t, x))$, $t = t_k$, $x \in S_\psi$ *and the function* $\rho_k : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ *is nondecreasing for* $k = 1, 2, \ldots$

(iii) $r(t) = r(t; t_0, u_0) \in PC^q([t_0, T], \mathbb{R}^n)$ *is the maximal solution of the the IVP for the IFrDE* (6).

Then,

$$V(t, x(t)) \leq r(t), t \geq t_0 \tag{7}$$

where $x(t) = x(t; t_0, x_0) \in PC^q([t_0, T], \mathbb{R}^N)$ *is any solution of* (4) *existing on* $[t_0, \infty)$, *provided that*

$$V(t_0^+, x_0) \leq u_0. \tag{8}$$

Proof. Let $\eta \in \bar{S}_\psi =: \{\eta \in \mathbb{R}^n : \|\eta\| \leq \psi\}$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations.

$$\begin{aligned} {}^cD^q u = g(t, u) + \eta, \Delta u = \psi_k(u(t_k)), t = t_k, k = 1, 2, ... \\ u(t_0^+) = u_0 + \eta \end{aligned} \tag{9}$$

for $t \in [t_0, \infty)$.
The function $u_\eta(t, \alpha)$ is a solution of (9), where $\alpha > 0$, if and only if it satisfies the Volterra Integral equation

$$u_\eta(t, \alpha) = u_0 + \eta + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1}(g(s, u_\eta(s, \alpha)) + \eta)ds, \, t \in [t_0, \infty) \tag{10}$$

Let the function $m(t, \alpha) \in PC([t_0, T] \times \bar{S}_\psi, \mathbb{R}^N)$ be defined as $m(t, \alpha) = V(t, x^*(t))$.
We now prove that

$$m(t, \alpha) < u_\eta(t, \alpha), \quad \text{for} \quad t \in [t_0, \infty) \tag{11}$$

Observe that the inequality (11) holds for $t = t_0$ i.e

$$m(t_0, \alpha) = V(t_0, x_0) \leq u_0 < u_\eta(t_0, \alpha)$$

Assume that the inequality (11) is not true, then there exist a point $t_1 > t_0$ such that

$$m(t_1, \alpha) = u_\eta(t_1, \alpha) \quad \text{and} \quad m(t, \alpha) < u_\eta(t, \alpha) \quad \text{for} \quad t \in [t_0, t_1)$$

It follows from lemma (3.6) that

$${}^CD_+^q m(t_1, \alpha) - {}^C D_+^q u_\eta(t_1, \alpha) > 0$$

So that

$${}^CD_+^q(V(t_1, x(t_1))) > {}^C D_+^q(u_\eta(t_1, \alpha))$$

and using (9) we arrive at

$${}^CD_+^q(V(t_1, x(t_1))) > g(t_1, u_\eta(t_1, \alpha) + \eta) > g(t_1, u(t_1, \alpha))$$

Therefore,

$${}^CD_+^q(m(t_1, \alpha)) > g(t_1, u(t_1, \alpha)) \tag{12}$$

For $t \in [t_0, T]$, we maintain that $x^*(t)$ satisfies (4) and the equality,

$$\limsup_{h \to 0^+} \frac{1}{h^q}[x^*(t) - x_0 - S(x^*(t), h)] = f(t, x^*(t)) \tag{13}$$

holds, where $x^*(t)$ is any other solution of (4).

$$S(x^*(t), h) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[x^*(t - rh) - x_0)] \tag{14}$$

is the Grunwald Letnikov fractional derivative and $[\frac{t-t_0}{h}]$ is the integer part of $\frac{t-t_0}{h}$.

Multiply equation (13) althrough by h^q we have,

$$\limsup_{h\to 0^+}[x^*(t) - x_0 - S(x^*(t), h)] = h^q f(t, x^*(t))$$
$$x^*(t) - x_0 - \limsup_{h\to 0^+}[S(x^*(t), h)] = h^q f(t, x^*(t))$$
$$x^*(t) - x_0 - [S(x^*(t), h) + \rho(h^q)] = h^q f(t, x^*(t))$$

$$x^*(t) - h^q f(t, x^*(t)) = [S(x^*(t), h) + x_0 + \rho(h^q)] \tag{15}$$

For $t \in [t_0, T]$, we have

$$\tag{16}$$

$$m(t,\alpha) - m(t_0,\alpha) - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^qC_r)[m(t-rh,\alpha) - m(t_0,\alpha)]$$
$$= V(t, x^*(t)) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,q}C_r[V(t-rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)]$$
$$= V(t, x^*(t)) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,q}C_r[V(t-rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)]$$
$$+ \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^qC_r)[V(t-rh, S(x^*(t), h) + x_0 + \rho(h^q) - V(t_0, x_0)]$$
$$-[V(t-rh, x^*(t-rh) - V(t_0), x_0)] \tag{17}$$

Since $V(t,x)$ is locally Lipschtzian in the second variable, we have

$$\leq L|(-1)^{r+1}|||\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^qC_r)[S(x^*(t), h) + x_0 + \rho(h^q) - x^*(t-rh)]||$$

where $L > 0$ is the Lipschitz constant.

$$\leq L||\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r[S(x^*(t), h) + \rho(h^q) - (x^*(t-rh) - x_0)]|| \tag{18}$$

Using equations (14), equation (18) becomes,

$$\leq L||\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r(\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,q}C_r[x^*(t-rh) - x_0)] + \rho(h^q) - (x^*(t-rh) - x_0)]||$$

$$\leq L||\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r(-1)^{r+1}(\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r[x^*(t-rh) - x_0)] + \sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r\rho(h^q)$$
$$-\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r(x^*(t-rh) - x_0)]||$$

$$\leq L(-1)^{r+1}\|\sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(x^*(t-rh)-x_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r\rho(h^q)\| \tag{19}$$

Substituting equation (19) into (16) yields,

$$= V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t-rh, x^*(t)-h^q f(t,x^*(t)) - V(t_0,x_0)]$$

$$+L\|\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r(x^*(t-rh)-x_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r\rho(h^q)\|$$

$$= V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t-rh, x^*(t)-h^q f(t,x^*(t)) - V(t_0,x_0)]$$

$$+L\|\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r(x^*(t-rh)-x_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{rq}C_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r\rho(h^q)\|$$

Dividing althrough by $h^q > 0$ and taking the $limsup$ as $h \to 0^+$ we have,

$$\begin{aligned}
{}^CD_+^q m(t,\alpha) &= \limsup_{h\to 0^+} \frac{1}{h^q}[V(t,x^*(t)) - V(t_0,x_0) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t-rh, x^*(t)-h^q f(t,x^*(t)) - V(t_0,x_0)]] \\
&\quad + \limsup_{h\to 0^+} \frac{1}{h^q} L\|\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r(x^*(t-rh)-x_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{rq}C_r - 1] \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r\rho(h^q)\|
\end{aligned}$$

Recall $\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{rq}C_r = -1$ *and* $\limsup_{h\to 0^+}\frac{1}{h^q}\rho(h^q) = 0$ from equations (3.6) and (3.7) in [24] we have,

$${}^CD_+^q m(t,\alpha) = {}^C D_+^q V(t,x^*(t)) + L\|\sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(x^*(t-rh)-x_0)[-(-1)-1] + 0\|$$

$${}^CD_+^q m(t,\alpha) = {}^C D_+^q V(t,x^*(t)) + 0$$

Using condition (ii) of Theorem 4.1 we have

$${}^CD_+^q m(t,\alpha) \leq g(t, V(t,x^*(t))) = g(t, m(t,\alpha)) \tag{20}$$

Also,

$$m(t_0^+,\alpha) \leq u_0 \; and \; m(t_k^+,\alpha) = V(t_k^+, x(t_k) + I_k(x(t_k)) \leq \rho_k(m(t_k)) \tag{21}$$

Now, equation (20) with $t = t_1$ contradicts (12), hence (11) holds.

For $t \in [t_0, T]$, we now show that whenever $\eta_1 < \eta_2$, then

$$u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \tag{22}$$

It is obvious that (22) holds for $t = t_0$. Assume the inequality (22) is not true. Then there exist a point $t_1 > t_0$ such that $u_{\eta_1}(t_1,\alpha) = u_{\eta_2}(t_1,\alpha)$ and $u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha)$ for $t \in [t_0, t_1)$.
By lemma (3.6), we have that

$$^C D^q_+(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_1,\alpha)) > 0$$

However,

$$\begin{aligned}
^C D^q_+(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_1,\alpha_)) &= {}^C D^q_+ u_{\eta_1}(t_1,\alpha) - {}^C D^q_+ u_{\eta_2}(t_1,\alpha) \\
&= g(t_1, u(t_1,\alpha) + \eta_1)) - [g(t_1, u(t_1,\alpha) + \eta_2))] \\
&= \eta_1 - \eta_2 < 0
\end{aligned}$$

which is a contradiction and so (22) is true. Thus, equations (11) and (22) guarantee that the family of solutions $\{u_\eta(t,\alpha)\}, t \in [t_0, T]$ of (9) is uniformly bounded, i.e. there exists $P > 0$ with $|u_\eta(t,\alpha)| \leq P$, with bound P on $[t_0, T]$.
We now show that the family $\{u_\eta(t,\alpha)\}$ is equicontinuous on $[t_0, T]$. Assume $K = sup\{g(t,x) : (t,x) \in [t_0, T] \times [-P, P]\}$. Also, fix a decreasing sequence $\{\eta_i\}_{i=1}^{\infty}(t)$, such that $\lim_{i\to\infty} \eta_i = 0$ and consider a sequence of functions $u_{\eta_i}(t,\alpha)$. Again let $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$, then we have the following estimate

$$\begin{aligned}
\|u_{\eta_i}(t_2,\alpha) - u_{\eta_i}(t_1,\alpha)\| &= \|u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_2 - s)^{q-1}(g(s, u_{\eta_i}(s,\alpha)) + \eta_i) \\
&- (u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1 - s)^{q-1}(g(s, u_{\eta_i}(s,\alpha)) + \eta_i))\| \\
&= \frac{1}{\Gamma(q)}\|\int_{t_0}^{t_2}(t_2 - s)^{q-1}(g(s, u_{\eta_i}(s,\alpha)))ds \\
&- \int_{t_0}^{t_1}(t_1 - s)^{q-1}(g(s, u_{\eta_i}(s,\alpha)))ds\| \\
&\leq \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_2}(t_2 - s)^{q-1} - \int_{t_0}^{t_1}(t_1 - s)^{q-1}\right|ds \\
&= \frac{k}{\Gamma(q)}\left|-(\int_{t_0}^{t_1}(t_1 - s)^{q-1} - \int_{t_0}^{t_2}(t_2 - s)^{q-1})\right|ds \\
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1 - s)^{q-1} - (\int_{t_0}^{t_1}(t_2 - s)^{q-1} + \int_{t_1}^{t_2}(t_2 - s)^{q-1})\right|ds \\
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1 - s)^{q-1} - \int_{t_0}^{t_1}(t_2 - s)^{q-1} - \int_{t_1}^{t_2}(t_2 - s)^{q-1}\right|ds \\
&\leq \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1 - s)^{q-1} - \int_{t_0}^{t_1}(t_2 - s)^{q-1}\right|ds + \left|\int_{t_1}^{t_2}(t_2 - s)^{q-1}\right|ds \\
&= \frac{k}{\Gamma(q)}\left|\frac{(t_1 - t_0)^q}{q} + \frac{(t_2 - t_1)^q}{q} - \frac{(t_2 - t_0)^q}{q}\right| + \frac{(t_2 - t_1)^q}{q}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{k}{\Gamma(q+1)}(t_1 - t_0)^q + (t_2 - t_1)^q - (t_2 - t_0)^q + (t_2 - t_1)^q \\
&= \frac{k}{\Gamma(q+1)}(t_1 - t_0)^q - (t_2 - t_0)^q + 2(t_2 - t_1)^q \\
&\leq \frac{2k}{\Gamma(q+1)}(t_2 - t_1)^q < \epsilon
\end{aligned}$$

provided $\|t_2 - t_1\| < \delta = (\frac{\epsilon\Gamma(q+1)}{2k})^{\frac{1}{q}}$, proving that the family of solutions $\{u_{\eta_i}(t;\alpha)\}$ is equi-continuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t;\alpha)\}$ has a subsequence $\{u_{\eta_{i_j}}(t;\alpha)\}$ which converges uniformly to a function $r(t)$ on $[t_0, T]$. We then show that $r(t)$ is a solution of (10). Equation (10) becomes

$$u_{\eta_{i_j}}(t,\alpha) = u_{0_{i_j}} + \eta_{i_j} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g_{i_j}(s, u_{i_j}(s,\eta_{i_j})) + \eta_{i_j})ds \tag{23}$$

Taking the limit as $i_j \to \infty$ in (23), yields

$$r(t) = u_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g(s, r(t)))ds \tag{24}$$

Thus, $r(t)$ is a solution of (6) on $[t_0, T]$. We claim that $r(t)$ is the maximal solution of (6). Then from (11), we have that $m(t,\alpha) < u_\eta(t,\alpha) \le r(t)$ on $[t_0, T]$.

5 Main Results

In this section, we will obtain sufficient conditions for the uniform strict stability of the system (3.3).

Theorem 5.1. *Assume the following:*

(i) $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *satisfies* $(A_0)(ii)$ *and* $g(t,u)$ *is quasi-monotone non-decreasing in* u *with* $g(t,0) \equiv 0$.

(ii) $V : \mathbb{R}_+ \times S_\psi \to \mathbb{R}^N_+$, $V \in \mathbb{L}$ *is locally Lipschitzian in* x *with* $V(t,0) \equiv 0$ *such that*

$${}^{C}D^q_+ V(t,x) \le g(t, V(t,x)), t \ne t_k, (t,x) \in \mathbb{R}_+ \times S_\psi \tag{25}$$

holds for all $(t,x) \in \mathbb{R}_+ \times S_\psi$.

(iii) there exists a $\psi_0 > 0$ *such that* $x_0 \in S_\psi$ *implies that*

$x + I_k(x) \in S_\psi$ *and* $V(t, x + I_k(x)) \le \psi_k(V(t,x)), t = t_k, x \in S_\psi$ *and the function* $\psi_k : \mathbb{R}^N_+ \to \mathbb{R}^N_+$

is nondecreasing for k=1,2,...

(iv) $b(\|x\|) \le V_0(t,x) \le a(\|x\|)$, *where* $a, b \in \mathcal{K}$ *and* $V_0(t,x) = \sum_{i=1}^{N} V_i(t,x)$

Then the strict uniform stability of the trivial solution $u = 0$ *of the IFrDE (6) implies the strict uniform stability of the trivial solution* $x = 0$ *of (4).*

Proof. Let $0 < \epsilon < \psi$ and $t_0 \in R_+$ be given.
Assume that the solution $u = 0$ of (6) is strict uniformly stable. Then given each $b(\epsilon) > 0$, and $t_0 \in R_+$, there exist a positive function $\delta_1 = \delta_1(\epsilon) > 0$ such that whenever

$$u_0 = \sum_{i=1}^{n} u_{i0} < \delta, \ we \ \ have \ \sum_{i=1}^{n} u_i(t; t_0, u_0) \le b(\epsilon), t \ge t_0 \tag{26}$$

where $u(t; t_0, u_0)$ is any solution of (6).
let us choose $V(t_0^+, x_0) \le u_0$ and

$$\sum_{i=1}^{n} u_{i0} = a(t_0, \|x_0\|)$$

Since $a(t, \mathbb{K})$ and $a \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ we can find a positive function $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(t_0, \|x_0\|) < \delta_1 \ and \ \|x_0\| < \delta \tag{27}$$

hold simultaneously. We claim that if

$$\|x_0\| \leq \delta, \ then \ \|x(t, t_0, x_0)\| \leq \epsilon, t \geq t_0.$$

Suppose that this claim is not true. Then there would exists a point $t_1 > t_0$ and a solution $x(t)$ with $\|x_0\| < \delta$ such that

$$\|x(t_1)\| = \epsilon \ and \ \|x(t)\| < \epsilon, \ for \ t \in [t_0, t_1). \tag{28}$$

This implies that $x(t) + I_k(x) \in S_\psi \ for \ t \in [t_0, t_1)$.
From equation (7) we have that

$$V_0(t, x(t)) \leq r_0(t, t_0, u_0) \ for \ t \in [t_0, t_1). \tag{29}$$

Combining condition (iv) and (26) we have

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, x(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) \tag{30}$$

Using equations (26), (28) and (30) we have,

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, x(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) < b(\epsilon)$$

which leads to an absurdity that $b(\epsilon) < b(\epsilon)$.
Hence, the strict uniform stability of the trivial solution $u = 0$ of (6) implies the strict uniform stability of the trivial solution $x = 0$ of (4).

6 Application

Let the points $t_k, t_k < t_{k+1}$, $lim_{k\to\infty} t_k \to \infty$ be fixed. Consider the vector impulsive Caputo fractional differential equations

$$\begin{aligned} {}^C D^q x_1(t) &= 2x_1 sinx_1 + \frac{x_2^2 cosecx_2}{x_1} - 10x_1, t \neq t_k \\ {}^C D^q x_2(t) &= 3x_2 cosecx_2 + 4x_2 sinx_1 - \frac{x_1^2 secx_2}{x_2}, t \neq t_k \\ \Delta x_1 &= s_k(x(t_k)), \Delta x_2 = n_k(x(t_k)), t = t_k, k = 1, 2, ... \end{aligned} \tag{31}$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0^+) = x_{10} \quad \text{and} \quad x_2(t_0^+) = x_{20}$$

Consider a vector $V = (V_1, V_2)^T$, where
$V_1(t, x_1, x_2) = x_1^2$ and $V_2(t, x_1, x_2) = x_2^2$, with $x = (x_1, x_2) \in \mathbb{R}^2$, and its associated norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$.
Now

$$V_0(t, x) = \sum_{i=1}^{2} V_i(t, x_1, x_2) = x_1^2 + x_2^2$$

and so $b(\|x\|) \leq V_0(t,x) \leq a(\|x\|)$ with $b(r)=r$ and $a(r)=r^2$, implying that $a,b \in \mathcal{K}$. From (5), we compute the Caputo fractional Dini derivative for $V_1(t,x_1,x_2)=x_1^2$ for $t>0, t \neq t_k$ as follows:

$$\begin{aligned}
{}^{C}D_{+}^{q}V_1(t;x_1,x_2) &= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{V(t,x)-V(t_0,x_0)\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)[V(t-rh,x-h^q f(t,x))-V(t_0,x_0)]\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_1^2-x_{10}^2+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)[(x_1-h^q f_1(t;x_1,x_2)^2-x_{10}^2]\big\}\\
&\leq \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_1^2-x_{10}^2|\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)[x_1^2-2x_1h^q f_1(t;x_1,x_2)+h^{2q}f_1(t,x_1)-x_{10}^2]\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_1^2-x_{10}^2+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)x_1^2\\
&\quad-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)2x_1h^q f_1(t;x_1,x_2)\\
&\quad-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)h^{2q}f_1(t;x_1,x_2)\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_1^2+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)x_1^2-x_{10}^2-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)x_{10}^2\\
&\quad-2x_1\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)h^q f_1(t;x_1,x_2)|\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)x_1^2-\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)x_{10}^2\\
&\quad-2x_1\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{q}C_r)h^q f_1(t;x_1,x_2)\big\}
\end{aligned}$$

from equations (3.7) and (3.8) in [24].

$$\begin{aligned}
{}^{C}D_{+}^{q}V_1(t;x_1,x_2) &\leq \frac{x_1^2}{t^q\Gamma(1-q)}-\frac{x_{10}^2}{t^q\Gamma(1-q)}+2x_1f_1(t;x_1,x_2)\\
{}^{C}D_{+}^{q}V_1(t;x_1,x_2) &\leq \frac{x_1^2}{t^q\Gamma(1-q)}+2x_1f_1(t;x_1,x_2)\\
{}^{C}D_{+}^{q}V_1(t;x_1,x_2)&\\
&\leq \frac{x_1^2}{t^q\Gamma(1-q)}+2x_1(2x_1 sinx_1+\frac{x_2^2 cosecx_2}{x_1}-10x_1)\\
&\leq \frac{x_1^2}{t^q\Gamma(1-q)}+4x_1^2 sinx_1+2x_2^2 cosecx_2-20x_1^2
\end{aligned}$$

As $t \to \infty$, $\frac{x_1^2}{t^q\Gamma(1-q)} \to 0$, so that we have

$$
\begin{aligned}
&\leq +4x_1^2 sinx_1 + 2x_2^2 cosecx_2 - 20x_1^2 \\
&\leq x_1^2(-20 + 4|sinx_1|) + x_2^2(2\frac{1}{|sinx_2|}) \\
&\leq x_1^2(-16) + x_2^2(2) \\
&\leq -16V_1 + 2V_2
\end{aligned}
$$

Therefore,

$$ {}^C D_+^q V_1(t; x_1, x_2) \leq -16V_1 + 2V_2 \tag{32} $$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$ V(t, x(t) + s_k) = |s_k + x(t)| \leq V(t, x(t)) $$

Similarly, using (5), we compute the Caputo fractional Dini derivative for $V_2(t, x_1, x_2) = x_2^2$ as follows:

$$
\begin{aligned}
{}^C D_+^q V_2(t; x_1, x_2) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{V(t,x) - V(t_0, x_0) \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[V(t - rh, x - h^q f(t, x)) - V(t_0, x_0)]\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[(x_2 - h^q f_2(t; x_1, x_2)^2 - x_{20}^2]\} \\
&\leq \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2| \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[x_2^2 - 2x_2 h^q f_2(t; x_1, x_2) + h^{2q} f_2(t, x_2) - x_{20}^2]\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_2^2 \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)2x_2 h^q f_2(t; x_1, x_2) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)h^{2q} f_2(t; x_1, x_2)\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_2^2 - x_{20}^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_{20}^2 \\
&\quad -2x_2 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)h^q f_2(t; x_1, x_2)|\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{ \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_2^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_{20}^2
\end{aligned}
$$

$$
\begin{aligned}
& -2x_2 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_2(t;x_1,x_2)\} \\
{}^C D_+^q V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} - \frac{x_{20}^2}{t^q\Gamma(1-q)} + 2x_2 f_2(t;x_1,x_2) \\
{}^C D_+^q V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} + 2x_2 f_2(t;x_1,x_2) \\
{}^C D_+^q V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} + 2x_2(-3x_2 cosecx_2 + 4x_2 sinx_1 - \frac{x_1^2 secx_2}{x_2}) \\
&\leq \frac{x_1^2}{t^q\Gamma(1-q)} - 6x_2^2 cosecx_2 + 4x_2^2 sinx_1 + 2x_1^2 secx_2 \\
&\leq x_2^2(-6cosecx_2 + 4sinx_1) + x_1^2(2secx_2) \\
&\leq x_2^2(-6\frac{1}{|sinx_2|} + 4|sinx_1|) + x_1^2(\frac{2}{|cosx_2|}) \\
&\leq x_2^2(-6+4) + x_1^2(2) \\
&\leq x_2^2(-2) + x_1^2(2) \\
&\leq -2V_2 + 2V_1
\end{aligned}
$$

Therefore,

$$ {}^C D_+^q V_1(t;x_1,x_2) \leq 2V_1 - 2V_2 \tag{33} $$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$ V(t, x(t) + n_k) = |n_k + x(t)| \leq V(t, x(t)) $$

Combining (33) and (32), we have

$$ {}^C D_+ V \leq \begin{pmatrix} -16 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = g(t, V) \tag{34} $$

Now consider the comparison system

$$ {}^C D^q u = g(t, u) = Au \tag{35} $$

where $A = \begin{pmatrix} -16 & 2 \\ 2 & -2 \end{pmatrix}$.

The vectorial inequality (34) and all other conditions of Theorem (5.1) are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (35) is strictly uniformly stable . Therefore, the trivial solution $x_0 = 0$ of the system of IFrDE (31) is strictly uniformly stable.

7 Conclusion

In this study, we have explored the strictly uniform stability of the trivial solution for nonlinear ICFrDEs using the framework of vector Lyapunov functions. By introducing a generalized class of piecewise continuous Lyapunov functions and employing comparison results, we derived the strictly uniform stability of the trivial solution of the fractional dynamical systems. The provided example not only validates the theoretical results but also extends and refines existing findings, demonstrating the utility and versatility of the proposed approach. This work contributes a significant step forward in the stability analysis of fractional impulsive systems and open avenues for further research into more complex systems and applications.

Conflict of Interests: The authors declare that there is no conflict of interests.

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