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Options pricing using combination of Brownian motion with a random time component

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Abstract

This paper provides an efficient option pricing technique for European call options. The technique is a process that combines Geometric Brownian motion with random time component. The additional parameter to the Geometric Brownian motion is the volatility of the time change. This additional parameter provides control over the skewness and kurtosis of the return distribution. Partial integro-differential equation (PIDE) for pricing options under the proposed process was derived. Its values were priced numerically and the pricing performance of the new model is compared on option data sourced from Bloomberg on 28th April, 2023 with the Geometric Brownian motion. The result revealed that the additional parameter was able to address the pricing biases of the Geometric Brownian motion. Recommendation was made that more work should be done in the advancement of quick and effective procedures for calculating PIDE's having in view to calibrating stochastic processes to a surface of option prices.

Keywords and phrases: Extended Brownian motion; Geometric Brownian motion; Lévy processes

1 Introduction

Geometric Brownian Motion (GBM) is a mathematical technique used to model the unpredictable movement of stock prices. It is widely used for stock price forecasting. This paper develops an efficient technique for valuing options when the basic asset price has dynamics given by a pure jump infinity activity Lévy process. The proposed technique is a process that combines Geometric Brownian motion with a random time component, governed by a gamma process. This approach allows for flexible modelling of economic time periods, where each unit of time has a variable length, described by a gamma distribution with unit mean and positive variance. The process yields normally distributed returns over a unit period, conditional on the random time realization, which follows a gamma distribution. This framework provides a strong three-parameter model, extending the existing Geometric Brownian motion model by incorporating additional parameter that controls kurtosis and skewness in the return distribution. We termed the proposed technique Extended Geometric

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Brownian Motion model (EGBM). The process has finite moments, distinguishing it from many Lévy processes. There is no diffusion component in the new process and it is thus a pure jump process.

The motivation for this study is to address the pricing biases of the Geometric Brownian motion of assuming constant volatility, which is not realistic as volatility changes over time and assuming a log-normal distribution of asset prices, which may not capture extreme events making it less sensitive to changes in underlying variables and less suitable for modelling assets with volatile prices. The fact that the price change of the asset is often influenced by the suddenly incoming and highly important price-making information, which results rather in the significant price jump makes it necessary to develop an efficient technique for valuing options and a model that could more accurately, express the empirical leptokurtic distributions of logarithmic returns in financial assets. Models that are incorporated with jumps offer better mathematical tool as described in [1], [2], [3], [4], [5], [6], [7]. We used three studies to demonstrate the effectiveness of Geometric Brownian motion in this paper. [8] tested the Geometric Brownian motion model for predicting stock prices in Nigeria. The model worked well for short-term predictions but was less accurate for long-term prediction. [9] used GBM to predict short-term stock prices for 24 Malaysian stocks, achieving accurate results for up to two weeks. [10] simulated 50 Australian stock prices using GBM and the Capital Asset Pricing Model (CAPM), finding that the model's predictions matched actual price movements around 50% of the time, across various timeframes. These suggest that GBM is a reliable tool for short-term stock price forecasting, with moderate accuracy over different time horizons. However, GBM does not predict extreme market events or crashes. It assumes constant volatility and continuous trading, which may not accurately reflect real-world market conditions. GBM also assumes continuous price movements, but in reality, prices can exhibit sudden jumps. We derived a form of partial integro-differential equation (PIDE) particularly suited to our proposed numerical implementation. This result is followed by developing a numerical method that will compute an approximate solution to the PIDE. However, we claim that the method is consistent, stable, and converging. This claim is built on the observation that the numerical values for European options converge to the closed form solutions presented in [11]. We calibrated the observed values of the market for European options prices sourced from Bloomberg on 28th April, 2023.

2 Methodology/Model formulation

This section presents the methods adopted in this paper.

2.1 Geometric Brownian Motion (GBM)

Given $\theta, \sigma > 0$ and $x_0 > 0$. The GBM is the continuous time process, X_t that solves the SDE

$$dS_t = \theta S_t dt + \sigma S_t dW_t \tag{1}$$

Here, S is the stock price, θ and σ are constants, t is time, and W follows a stochastic process called Wiener process, under which $dW = \varepsilon\sqrt{dt}$, where ε is a random draw from the standardized normal distribution. (1) is commonly known as Geometric Brownian motion (GBM), with θ and σ called the drift parameter and the volatility parameter, respectively. The solution is given by

$$d\ln S = \left(\theta - \frac{\sigma^2}{2}\right)dt + \sigma dW \tag{2}$$

The stochastic process as characterized by (2) indicates that $\ln S$ is normally distributed. Equivalently, S is lognormally distributed. With S_0 and S_T denoted as the stock prices at time 0 and time T, respectively, (2) leads to

$$S_T = S_0 exp\left[\left(\theta - \frac{\sigma^2}{2}\right)T + \sigma\varepsilon\sqrt{T}\right] \tag{3}$$

Further, the expected value and the variance of S_T are

$$E\left(S_T\right) = S_0 \exp\left(\theta T\right) \tag{4}$$

and

$$Var\left(S_T\right) = S_0^2\left[\exp\left(2\theta T\right)\right]\left[\exp\left(\sigma^2 T\right) - 1\right] = \left[E\left(S_T\right)\right]^2\left[\exp\left(\sigma^2 T\right) - 1\right] \tag{5}$$

2.2 The Extended Geometric Brownian Motion Model (EGBM)

The EGBM process $X\left(t;\sigma,\nu,\theta\right)$ is derived by approximating a Brownian motion $W\left(t\right)$ with drift $b\left(t;\theta,\nu\right) = \theta t + \sigma W\left(t\right)$ at a random time given by a gamma process with unit mean rate, $\gamma\left(t;1,\nu\right)$,

$$X\left(t;\sigma,\nu,\theta\right) = b\left(\gamma\left(t;1,\nu\right);\theta,\sigma\right). \tag{6}$$

The (EGBM) process consist of three parameters: first is the volatility σ of the Brownian motion, followed by the drift θ of the Brownian motion, then the volatility ν of the time change which is the new parameter added to the existing Geometric Brownian Motion.
The EGBM process has its characteristic function as:

$$\phi X\left(t\right)^{(u)} = \mathrm{E}\left(e^{iuX(t)}\right) = \left(\frac{1}{1 - iu\theta\nu + \sigma^2u^2/2}\right)^{\frac{t}{\nu}} \tag{7}$$

This characteristic function is infinitely divisible with a zero continuous martingale component and a Lévy measure denoted by

$$k\left(x\right)dx = \frac{\theta x/\sigma^2}{\nu/x}\,\mathrm{esp}\left(\frac{\sqrt{2/\nu + \theta^2/\sigma^2}}{\sigma}\left|x\right|\right)dx \tag{8}$$

2.3 The Extended Brownian Motion Model European Option Pricing Formulas

Let $S\left(t\right)$ be the stock price at time t. The EGBM risk neutral process for the stock price is denoted by

$$S\left(t\right) = S\left(0\right)e^{(r-q)t + X(t) + \omega t} \tag{9}$$

where the normalization factor $e^{\omega t}$ ensuring that $\mathrm{E}\left[S\left(t\right)\right] = S\left(0\right)e^{(r-q)t}$, requires that $\omega = \frac{1}{\nu}\ln\left(1 - \sigma^2\nu/2 - \theta\nu\right)$.

The price of a European call option with strike K and maturity T following the risk neutrality is given as

$$c\left(S(0);K,t\right)=e^{-rt}\mathrm{E}_t\left(\left(S(t)-K(\ \ )\right)^+\right) \tag{10}$$

The Derivation of the Partial Integro-Differential Equation (PIDE)

The partial integro-differential equation was derived following the work of [12]

Suppose, $Y(t)=\log\left(S(t)\right)$. This follows from (9) that

$$Y(t)=Y(0)+(r-q+\omega)t+X\left(t;\sigma,\nu,\theta\right).$$

Note, that $X\left(t;\sigma,\nu,\theta\right)$ is a jump process that may be written as the sum of all its jumps we may write the explicit semi-martingale decomposition for $Y(t)$ as follows.

Let μ be the value of the random measure associated with the jumps of the process $X\left(t;\sigma,\nu,\theta\right)$ and let ν denotes the predictable compensator. Considering the specific case of EGBM we have that

$$\nu(dx,dt)=k(x)dxdt$$

$$k(x)=\frac{e^{-\lambda px}}{\nu x}1_{x>0}+\frac{e^{-\lambda_n|x|}}{\nu|x|}1_{x<0}$$

$$\lambda_p=\left(\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}}-\frac{\theta}{\sigma^2}$$

$$\lambda_n=\left(\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}}+\frac{\theta}{\sigma^2}$$

The semi-martingale decomposition for $Y(t)$ is given by

$$Y(t)=Y(0)+(r-q+\omega)t+x*\nu t+x*(\mu-\nu)t$$

where the operation $*$ signifies a double integration with respect to x, t in the domain

$\mathbb{R}\times[0,t]$, and $\mathbb{R}$ denotes the real line.

The infinitesimal generator for the Markov process $Y(t)$ applies to functions *f(Y, t)* $f(Y,t)$ and given by

$$\begin{aligned}\Gamma f=&\left[\frac{\partial}{\partial t}+\left(r-q+\omega+\int_{-\infty}^{\infty}xk(x)dx\right)\frac{\partial}{\partial Y}\right]f+\\&\int_{-\infty}^{\infty}\left[f(Y+x,t)-f(Y,t)-x\frac{\partial}{\partial Y}f\right]k(x)dx\end{aligned} \tag{11}$$

We deduce and prove that the option pricing PIDE in the Lévy models is of the form:

$$\begin{aligned}&\frac{\partial\omega}{\partial\tau}(x,\tau)-(r-q+\omega)\frac{\partial\omega}{\partial x}(x,\tau)+r\omega(x,\tau)\\&-\int_{-\infty}^{+\infty}\left[\omega(x+y,\tau)-\omega(x,\tau)\right]k(y)dy\\&-1_{x<x(\tau)}\left\{rK-qe^{x}-\int_{x(\tau)-x}^{\infty}\left[\omega(x+y,\tau)-\left(K-e^{x+y}\right)\right]k(y)dy\right\}=0\end{aligned} \tag{12}$$

This PIDE must be solved subject to initial condition

$$\omega(x,0)=\left(K-e^{x}\right)^{+}, \tag{13}$$

and boundary conditions

$$\frac{\partial^{2\omega}}{\partial x^{2}}(-\infty,\tau)-\frac{\partial\omega}{\partial x}(-\infty,\tau)=0\ \forall\tau \tag{14}$$

$$\frac{\partial^{2\omega}}{\partial x^{2}}(+\infty,\tau)-\frac{\partial\omega}{\partial x}(+\infty,\tau)=0\ \forall\tau \tag{15}$$

3 Analysis of the model

3.1 Discretization

To compute the numerical solution, we employed the finite difference discretization scheme developed and analyzed by [13]. This scheme utilizes a uniform spatial discretization with a step size of $\Delta x = 0.01$ and implicit time discretization with a step size of $\Delta t = 0.005$. We selected $N = 400$ spatial discretization steps and $M = 200$-time discretization steps. The spatial computational domain was limited to $x \in [-L, L]$, where $L = 4$. The rationale behind choosing this scheme is its computational efficiency, as it requires less computational resources and time compared to other methods.

4 Numerical simulations/sensitivity analysis

In this section comparisons are made between the value of a European call option under the assumption that price movements follow Extended Geometric Brownian motion and the corresponding Geometric Brownian motion.

4.1 Calibration of Extended Geometric Brownian Motion Model to Real Market Prices

Here, we calibrate EGBM to data obtained from real market. The data used was extracted from Bloomberg. The data consists of option market data observed on NASDAQ (National Association of Securities Dealers Automated Quotations) index Call Option Price Quotations on 28th April,2023. We calibrated EGBM parameters for each maturity separately resulting to parameters in Table 1.

Table 1: Calibrated EGBM Parameters

T	S_0	r	q	σ	ν	θ
0.3397	268.74	0.0423	0.021	0.2687	0.2317	-0.4064
0.4164	268.74	0.0426	0.022	0.2650	0.4240	-0.3883
0.5547	268.74	0.0429	0.021	0.2707	0.6908	-0.3811

0.6316	268.74	0.0421	0.022	0.2872	0.7025	-0.3288

Note from Table 1 that as maturity T gets higher, the annualized kurtosis parameter ν gets larger and annualized skewness θ reduces. The increase in ν is slower than the increase in maturity T and this is stable with an approach to normality, though at a rate slower than would be the case if ν were constant. The decrease in θ is also stable with the approach to normality.

Table 2: Early Exercise Premiums for EGBM and GBM

Maturity	$T_1 = 0.3397$		$T_2 = 0.4164$		$T_3 = 0.5547$		$T_4 = 0.6316$	
Strike	**GBM**	**EGBM**	**GBM**	**EGBM**	**GBM**	**EGBM**	**GBM**	**EGBM**
100	0.0412	0.0422	0.0324	0.0520	0.353	0.429	0.656	1.032
120	0.0317	0.0361	0.1333	0.1320	0.745	0.776	0.861	1.250
140	0.0120	0.1317	0.1365	0.1910	0.569	0.835	1.131	1.489
160	0.0565	0.1559	0.0381	0.2530	0.918	0.997	1.192	1.742
180	0.1313	0.1396	0.2351	0.2440	0.856	1.185	1.483	2.023
200	0.1564	0.1239	0.3359	0.2400	1.298	1.388	1.756	2.339
220	0.2531	0.1328	0.8853	0.2956	1.318	1.681	1.966	2.741
240	0.3528	0.6108	0.5365	0.4791	1.856	2.008	2.462	3.178
260	0.5462	0.8169	0.6745	1.0780	1.999	2.392	2.686	3.687
280	0.630	1.1103	1.0669	1.3740	2.664	2.795	3.400	4.241
300	1.066	1.4183	1.4375	1.6780	3.112	3.291	3.771	4.861

Table 2 gives the early exercise premiums for pricing options under the Extended Geometric Brownian motion model and Geometric Brownian motion dynamics respectively. Note that for all strikes and maturities, the EGBM early exercise premiums dominate those from Geometric Brownian motion. The differences are significant, for example when strike price is 220 with maturity 0.4164, it is about a 1/3 of the option value under Geometric Brownian motion. This drove us into concluding that the pricing of options under the underlying dynamics may be difficult but more efficient, it is important from the viewpoint of accounting for the values of these instruments correctly. The Extended Brownian motion model does not require unrealistic assumptions and is more consistent with the financial theory.

4.2 Simulated Price Path Comparisons for the EGBM and GBM

This section gives a comparison of stock price paths for the EGBM and GBM. To give a meaningful comparison, we have ensured that the same random numbers and time are used to generate all the paths. The simulated price paths of EGBM and GBM models are shown in Figure 1.

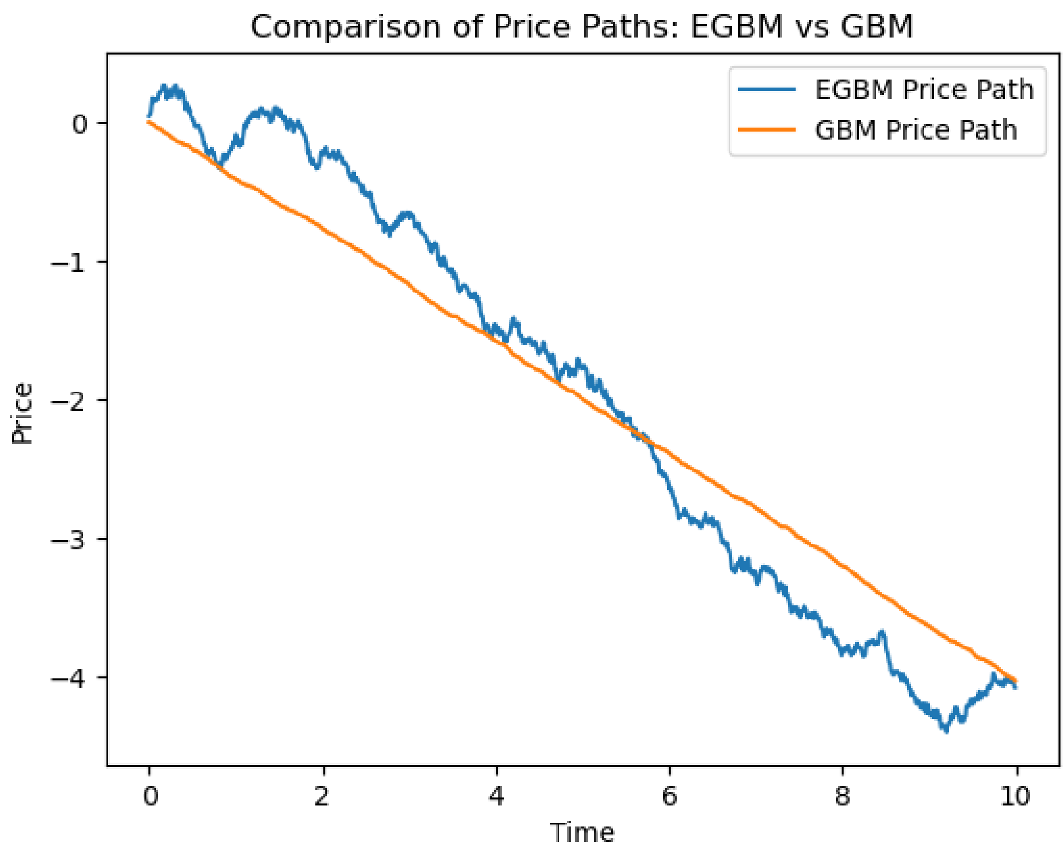

Figure 1: Simulated Price Paths of EGBM and GBM Models

It could be observed that the EGBM price path exhibits more volatility and randomness compared to the GBM price path, which is consistent with the EGBM model's stochastic volatility and jump components. It could also be noticed that the EGBM price path has a more irregular and unpredictable behavior, with sudden jumps and changes in direction, reflecting the model's ability to capture extreme events and volatility clustering. The GBM price path on the other hand, appears smoother and more predictable, with a more consistent and gradual change in price over time, reflecting the model's constant volatility and lack of jumps. The negative drift parameter θ **= -0.4064** in both models causes the price paths to decline over time, but the EGBM price paths declines more rapidly and unpredictably due to its stochastic volatility and jump component. The EGBM variance coefficient $\nu = 0.2317$ introduces additional randomness and variability making the EGBM price path more sensitive to changes in the underlying variables. The figure 1 demonstrates the EGBM ability to capture more complex and realistic market dynamics. However, the EGBM model is more capable of producing stock price returns distributions with fat tails and which exhibit skewness.

4.3 Volatility Paths of EGBM Model Compared with Volatility Paths of GB

This section shows the simulated volatility paths of EGBM and GBM models which are demonstrated in Figure 2

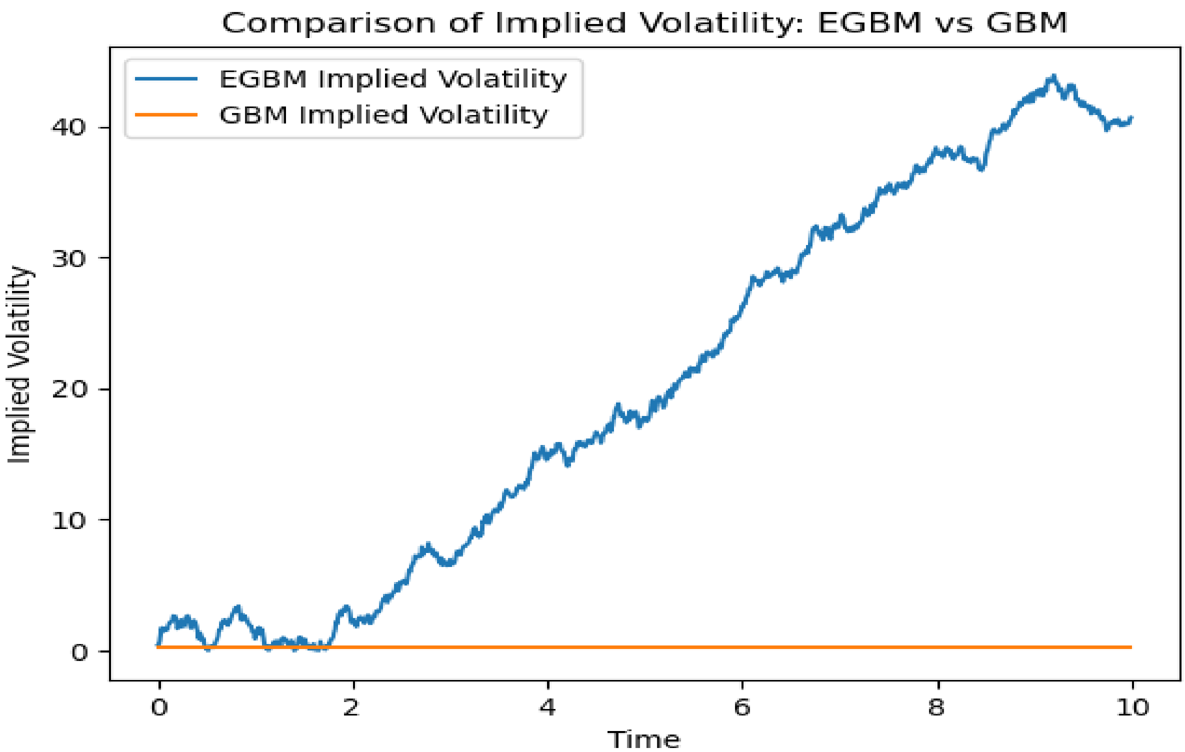

Figure 2: Volatility Paths of EGBM Model Compared with Volatility Paths of GBM Model

The EGBM model exhibits higher implied volatility than GBM indicating higher uncertainty or risk. The GBM model's constant implied volatility indicates a constant level of uncertainty or risk, whereas the EGBM model's time-varying implied volatility suggests a changing risk profile. Note that the implied volatility is a measure of the market's expected volatility, and the EGBM implied volatility is more volatile and unpredictable indicating a greater uncertainty in the market's expectations, hence demonstrating the EGBM ability to capture more unrealistic market dynamics.

5 Discussions and conclusion

We propose a model that combines Geometric Brownian motion with random time component. The proposed model is termed the Extended Geometric Brownian motion. Partial integro-differential equation (PIDE) for pricing options under the proposed process was derived and its values were estimated numerically. The pricing performance of the new model is compared on option data sourced from Bloomberg on 28th April, 2023 with the Geometric Brownian motion (GBM). The result revealed that the additional parameter was able to address the pricing biases of the Geometric Brownian motion. The approach developed here would consequently be applicable to a varied class of problems. However, more work is required in developing fast and efficient procedures for solving PIDE's with a view to calibrating stochastic processes to a surface of option prices.

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